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SOME GENERAL THEOREMS OF THE MECHANICS OF A DEFORMABLE SOLID

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There is given a formulation of quasistatic mixed problems of the mechanics of a deformable solid in displacements (problem A) and in stresses (problem B). There is presented an appropriate variational formulation of these problems on the basis of introducing a Lagrangian and a Castiglianian, and also the determination of the generalized solution of these problems. Under certain constraintes on the governing relations, there are proved theorems for the existence of the generalized solution of the problem A and its uniqueness, a theorem on the minimum of the Lagrangian, and also the convergence of successive approximations under the condition that the corresponding linear problem has a linear solution. There are considered methods to accelerate the convergence, including a "rapidly converging" successive approximations method having a substantially higher rate of convergence than a geometric progression. There is presented a new formulation of the quasistatic problem on the mechanics of a deformable solid in stresses (problem B), which converges to the solution of six equations in the stress tensor components with six boundary conditions. There is proved the equivalence of the formulation of the problem B to classical formulations.

1. In a certain Cartesian coordinate system, let the governing relationships connecting the stress tensor σ and the strain tensor ε be given in operator from [1]

$$\sigma_{ij} = F_{ij}(\varepsilon) \tag{1.1}$$

We consider the strains small so that the Cauchy relationships connecting them to the displacement vector are satisfied

$$\varepsilon_{ij} = {}^1/_2\,(u_{i,j} + u_{j,i}) \quad (\varepsilon = \operatorname{Def} \mathbf{u}) \tag{1.2}$$

Let the equilibrium equations of the medium be given ($\mathbf{X}$ are given volume forces) as well as boundary conditions of mixed type: the displacements $\mathbf{u}^\circ$ are given on the part Σ_1 of the body boundary, and the load S° on the other part Σ_2

$$\sigma_{ij,j} + X_i = 0 \quad u_i|_{\Sigma_1} = u_i^\circ, \quad \sigma_{ij}n_j|_{\Sigma_2} = S_i^\circ \tag{1.3}$$

We will consider all the functions under consideration to possess the smoothness needed to perform the manipulations used, and to vary in the time segment $[0, t_1]$, i.e., $0 \leqslant t \leqslant t_1$. Moreover, we shall assume the presence of a "natural" state i.e., we shall consider that the strain and stress tensors together with all their derivatives are zero in the time preceding $t = 0$. We consider the operator (1.1) local in the coordinates $\mathbf{x}$.

Substituting (1.2) into (1.1) and the result into (1.3), we obtain a system of three equations in the displacement vector components $\mathbf{u}$ with given boundary conditions

$$\sigma_{ij,j}(\mathbf{u}) + X_i = 0 \qquad (1.4)$$
$$u_i|_{\Sigma_1} = u_i^\circ, \quad \sigma_{ij}(\mathbf{u})\, n_j|_{\Sigma_2} = S_i^\circ$$

The abbreviated writing $\sigma_{ij}(\mathbf{u})$ means the following:

$$\sigma_{ij}(\mathbf{u}) \equiv F_{ij}(\varepsilon(\mathbf{u}))$$

where $\varepsilon_{ij}(\mathbf{u})$ are determined by the relationships (1.2).

A formulation of the quasistatic (static) problem of the mechanics of a deformable solid is given in displacements (problem A) by the relationships (1.1)–(1.3).

Let us multiply (1.4) scalarly by the as yet arbitrary vector $\mathbf{v}$ and let us integrate over the volume V occupied by the body. Then by using the Ostrogradskii-Gauss theorem [1] and the static boundary conditions in (1.4), we obtain

$$\int_V \sigma_{ij}\varepsilon_{ij}(\mathbf{v})\, dV = A^e(\mathbf{v}) + A_{\Sigma_1}(\mathbf{v}) \qquad (1.5)$$
$$A^e(\mathbf{v}) \equiv \int_V X_i v_i\, dV + \int_{\Sigma_2} S_i^\circ v_i\, d\Sigma$$
$$A_{\Sigma_1}(\mathbf{v}) \equiv \int_{\Sigma_1} \sigma_{ij} n_j v_i\, d\Sigma$$

where $A^e(\mathbf{v})$ is the work of the external forces in displacements $\mathbf{v}$, and $A_{\Sigma_1}(\mathbf{v})$ is the work of the internal forces at a given displacement $\mathbf{v}$.

Let us call the arbitrary vector field $\mathbf{v}(\mathbf{x}, t)$ a kinematic system, and the arbitrary field of second-rank symmetric tensors $\tau(\mathbf{x}, t)$ a static system. A kinematic system satisfying the kinematic boundary conditions in (1.3) is called kinematically admissible. We will write

$$\mathbf{v} \in U, \quad \text{if} \quad v_i|_{\Sigma_1} = u_i^\circ$$

A system satisfying the equilibrium equations (1.3) and static boundary conditions is called statically admissible. We will write

$$\tau \in T, \quad \text{if} \quad \tau_{ij,j} + X_i = 0, \quad \tau_{ij} n_j|_{\Sigma_2} = S_i^\circ$$

The difference between two kinematically admissible systems satisfies the homogeneous kinematic boundary conditions

$$\mathbf{v} \in U_0, \quad \text{if} \quad v_i|_{\Sigma_1} = 0 \qquad (1.6)$$

and the difference between two statically admissible systems satisfies the homogeneous equilibrium equations and the homogeneous static boundary conditions

$$\tau \in T_0, \quad \text{if} \quad \tau_{ij,j} = 0, \quad \tau_{ij} n_j|_{\Sigma_2} = 0 \qquad (1.7)$$

It follows from (1.6) and (1.5) that for the function $\mathbf{v}(\mathbf{x}) \in U_0$, from (1.4), the expression below is valid

$$\int_V \sigma_{ij}\varepsilon_{ij}(\mathbf{v})\, dV = A^e(\mathbf{v}) \qquad (1.8)$$

2. Let us call the function $\mathbf{u} \in U$ for which the Cauchy relations (1.2) and the governing equations (1.1) are valid and which satisfies the identities (1.8) for

every sufficiently smooth function $\mathbf{v} \in U_0$ the generalized solution of problem A. In other words, a function $\mathbf{u} \in U$ satisfying the integral identity

$$\int_V \sigma_{ij}(\mathbf{u})\, \varepsilon_{ij}(\mathbf{v})\, dV = A^e(\mathbf{v}) \tag{2.1}$$

for every smooth function $\mathbf{v} \in U_0$ is called a generalized solution of problem A.

It has been shown above that the solution of problem A is also its generalized solution.

T h e o r e m 2.1. If the generalized solution is sufficiently smooth, then it is a solution of problem A.

In fact, the solution of problem A should satisfy conditions (1.1)–(1.3). By the definition of a generalized solution, the relationships (1.1) and (1.2) and the first of the boundary conditions in (1.3) are satisfied. Applying the Ostrogradskii-Gauss theorem to the identity (1.8), we obtain

$$\int_V (\sigma_{ij,\,j} + X_i)\, v_i\, dV - \int_{\Sigma_2} (\sigma_{ij} n_j - S_i^\circ)\, v_i\, d\Sigma = 0 \tag{2.2}$$

Because of arbitrariness of the field $\mathbf{v} \in U_0$, the equilibrium equations and the static boundary conditions (1.3) hence follow.

Now, let us assume that the stress tensor is derivable from a potential. This means that there exists a scalar operator of the strain $W(\boldsymbol{\varepsilon})$ such that

$$\sigma_{ij} = F_{ij}(\boldsymbol{\varepsilon}) = \frac{\partial W(\boldsymbol{\varepsilon})}{\partial \varepsilon_{ij}} \tag{2.3}$$

A functional derivative is meant here, which is determined as follows, together with the differential Df of the operator $f(\mathbf{a})$:

$$Df\{\mathbf{a}, \mathbf{b}\} \equiv \left[\frac{\partial f}{\partial a_{ij}} b_{ij}\right] \equiv \frac{d}{d\xi} f(\mathbf{a} + \xi \mathbf{b})\,|_{\xi=0}$$

where ξ is a numerical parameter.

If the relationships (2.3) are valid and the mass and surface forces possess a potential, then the "Lagrangian" L can be introduced by means of the formula

$$L(\mathbf{u}) \equiv \Phi(\mathbf{u}) - A^e(\mathbf{u}), \quad \Phi(\mathbf{u}) \equiv \int_V W\, dV \tag{2.4}$$

The identity (2.1) can then evidently be written in the form

$$DL\{\varepsilon(\mathbf{u}), \ \varepsilon(\mathbf{v})\} \equiv D\{\mathbf{u}, \mathbf{v}\} = 0$$

Thus, the problem of seeking a generalized solution of the problem A is equivalent to the problem of seeking the "stationary point" of the Lagrangian $L(\mathbf{u})$.

If the relationships (2.3) are sufficiently smooth, then functional derivatives of the type

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}(\boldsymbol{\varepsilon}(\mathbf{u})) \equiv \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}(\mathbf{u}) \tag{2.5}$$

can be constructed,

L e m m a 1. If functional derivatives (2.5) of the governing relations (2.3) exist, then the following identity is valid

$$\Phi(\mathbf{u}_2) = \Phi(\mathbf{u}_1) + A^e(\mathbf{u}_2 - \mathbf{u}_1) + \frac{1}{2}\int_V \frac{\partial\sigma_{ij}}{\partial\varepsilon_{kl}}\{\mathbf{u}_1 + \eta(\mathbf{u}_2 - \mathbf{u}_1)\}[\varepsilon_{kl}(\mathbf{u}_2) - \varepsilon_{kl}(\mathbf{u}_1)] \times [\varepsilon_{ij}(\mathbf{u}_2) - \varepsilon_{ij}(\mathbf{u}_1)]\,dV \tag{2.6}$$

In fact, we introduce a function of the numerical argument ξ

$$f(\xi) \equiv \Phi\{\mathbf{u}_1 - \xi(\mathbf{u}_2 - \mathbf{u}_1)\},\ 0 \leqslant \xi \leqslant 1 \tag{2.7}$$

which admit the following representation on the segment mentioned

$$f(1) = f(0) + f'(0) + {}^1/_2 f''(\eta),\quad 0 < \eta < 1 \tag{2.8}$$

Substituting the expresssions obtained from (2.7) into (2.8) and taking account of(2.3) and (2.4), we obtain

$$\Phi(\mathbf{u}_2) = \Phi(\mathbf{u}_1) + \int_V \sigma_{ij}(\mathbf{u}_1)[\varepsilon_{ij}(\mathbf{u}_2) - \varepsilon_{ij}(\mathbf{u}_1)]\,dV + \frac{1}{2}\int_V \frac{\partial\sigma_{ij}}{\partial\varepsilon_{kl}}\{\mathbf{u}_1 + \eta(\mathbf{u}_2 - \mathbf{u}_1)\}[\varepsilon_{kl}(\mathbf{u}_2) - \varepsilon_{kl}(\mathbf{u}_1)][\varepsilon_{ij}(\mathbf{u}_2) - \varepsilon_{ij}(\mathbf{u}_1)]\,dV$$

Taking account of (1.8), we hence obtain (2.6).

T h e o r e m 2.2. Let us assume that the governing equations (1.1) are such that for each symmetric tensor of the second rank $\mathbf{h}$ the following inequality is satisfied

$$\left[\frac{\partial\sigma_{ij}}{\partial\varepsilon_{kl}} h_{kl}\right] h_{ij} \geqslant m_0 h_{ij} h_{ij},\quad m_0 > 0 \tag{2.9}$$

Then the stationary point of the Lagrangian (2.5) has a minimum.

Indeed, by setting $\mathbf{u}_2 = \mathbf{v} \in U_0$ in the identity (2.6), and $\mathbf{u}_1 = \mathbf{u}^*$, where $\mathbf{u}^*$ is a solution of the problem A, we have, by taking (2.9) into account,

$$L(\mathbf{v}) \equiv \Phi(\mathbf{v}) - A^e(\mathbf{v}) \geqslant \Phi(\mathbf{u}^*) - A^e(\mathbf{u}^*) + \frac{m_0}{2}\int_V \varepsilon_{ij}(\mathbf{v} - \mathbf{u}^*)\,\varepsilon_{ij}(\mathbf{v} - \mathbf{u}^*)\,dV \geqslant \Phi(\mathbf{u}^*) - A^e(\mathbf{u}^*) \equiv L(\mathbf{u}^*)$$

QED.

T h e o r e m 2.3. If the conditions (2.9) are satisfied, then there exist not more than one generalized solution of problem A.

Let us assume the opposite: there exist solutions $\mathbf{u}_1$ and $\mathbf{u}_2$. Then it follows from (2.1) that they satisfy the identity

$$\int_V [\sigma_{ij}(\mathbf{u}_2) - \sigma_{ij}(\mathbf{u}_1)]\,\varepsilon_{ij}(\mathbf{v})\,dV \tag{2.10}$$

Furthermore

$$[\sigma_{ij}(\mathbf{u}_2) - \sigma_{ij}(\mathbf{u}_1)]\,\varepsilon_{ij}(\mathbf{v}) = \int_0^1 \frac{\partial\sigma_{ij}}{\partial\varepsilon_{kl}}\{\mathbf{u}_1 + \xi(\mathbf{u}_2 - \mathbf{u}_1)\}[\varepsilon_{kl}(\mathbf{u}_2) - \varepsilon_{kl}(\mathbf{u}_1)]\,\varepsilon_{ij}(\mathbf{v})\,d\xi \tag{2.11}$$

Hence, setting $\varepsilon_{ij}(\mathbf{v}) \equiv \varepsilon_{ij}(\mathbf{u}_2) - \varepsilon_{ij}(\mathbf{u}_1)$, we obtain from (2.9)

$$0 \geqslant \int_V [\sigma_{ij}(\mathbf{u}_2) - \sigma_{ij}(\mathbf{u}_1)] [\varepsilon_{ij}(\mathbf{u}_2) - \varepsilon_{ij}(\mathbf{u}_1)] \, dV \geqslant$$
$$m_0 \int_V \varepsilon_{ij}(\mathbf{u}_2 - \mathbf{u}_1) \varepsilon_{ij}(\mathbf{u}_2 - \mathbf{u}_1) \, dV$$

It hence follows that

$$\varepsilon_{ij}(\mathbf{u}_2) \equiv \varepsilon_{ij}(\mathbf{u}_1)$$

i.e., the fields $\mathbf{u}_1(\mathbf{x})$ and $\mathbf{u}_2(\mathbf{x})$ can differ only by the displacement as a rigid whole. However, because of the first of the boundary conditions in (1.4), such displacements are not admissible. Hence, the uniqueness of the solution of problem A follows.

T h e o r e m 2 . 4. The point of the Lagrangian minimum is unique. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two points of the minimum of the functional L. Then condition (2.1) is satisfied for them and $\mathbf{u}_2 \equiv \mathbf{u}_1$ because of Theorem 2.3.

3. Now let us consider a certain linear tensor operator of the strain

$$p_{ij} = P_{ij}(\boldsymbol{\varepsilon}) \tag{3.1}$$

such that the quantity

$$(\mathbf{u}, \mathbf{v})_p \equiv \int_V p_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v}) \, dV \tag{3.2}$$

satisfies all scalar product axioms [2] in the functional space $\mathbf{u} \in U_0$ such that the functional space Π under consideration is a Hilbert space. Moreover, let the operator (3.1) be such that the inequalities

$$m p_{ij}(\mathbf{h}) h_{ij} \leqslant \left[\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} h_{kl}\right] h_{ij} \leqslant M p_{ij}(\mathbf{h}) h_{ij}, \quad 0 < m \leqslant M \tag{3.3}$$

are satisfied for the arbitrary symmetric tensor $\mathbf{h}$.

Let us note that if

$$p_{ij}(\boldsymbol{\varepsilon}) \equiv {}^1/_2 (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \varepsilon_{kl} = \varepsilon_{ij}$$

then the first of the inequalities (3.3) is equivalent to the inequality (2.9) for $m = m_0$, where the fact that the space Π is a Hilbert space follows in this case from the Korn inequalities [3].

Now, if a unique generalized solution of the problem A exists for the case when the operator of the governing relationships (1.1) is the operator P_{ij} of (3.1) (the problem A_p), successive approximations method can be constructed

$$p_{ij,j}(\mathbf{u}^{(n+1)}) = p_{ij,j}(\mathbf{u}^{(n)}) - \beta^{(n)} [\sigma_{ij,j}(\mathbf{u}^{(n)}) + X_i] \tag{3.4}$$
$$u_i^{(n+1)}\big|_{\Sigma_1} = u_i^\circ, \quad p_{ij}(\mathbf{u}^{(n+1)}) n_j \big|_{\Sigma_2} =$$
$$p_{ij}(u^{(n)}) n_j \big|_{\Sigma_2} - \beta^{(n)} [\sigma_{ij}(u^{(n)}) n_j \big|_{\Sigma_2} - S_i^\circ]$$

starting with some zero approximation $\mathbf{u}^{(0)}$ and be setting $n = 0, 1, \ldots$.

T h e o r e m 3 . 1. Let a unique generalized solution of the problem A_p exist, conditions (3.3) be valid, the volume and surface forces belong to the spaces L_q [2], where [4]

$$X \in L_q(V), \quad q > 6/5; \quad S^\circ \in L_q(\Sigma_2), \quad q > 4/3 \tag{3.5}$$

Moreover, let the condition

$$[\sigma_{ij}(\mathbf{u}^{(0)}) - p_{ij}(\mathbf{u}^{(0)})]\, h_{ij} \leqslant m p_{ij}(\mathbf{h})\, h_{ij} \tag{3.6}$$

be satisfied for the zero approximation $\mathbf{u}^{(0)}$ for an arbitrary symmetric tensor $\mathbf{h}$
Then in a certain neighborhood

$$\| \mathbf{u} - \mathbf{u}^{(0)} \|_p \leqslant r \tag{3.7}$$

there exists a generalized solution $\mathbf{u}^*$ of the problem A which is unique in this neighborhood, and for any value of the iteration parameter $\beta \in (0, 2/M]$ the successive approximations process (3.4) starting with $\mathbf{u}^{(0)}$ converges to it, where

$$\| \mathbf{u}^{(n)} - \mathbf{u}^{(0)} \|_p \leqslant q^n \| \mathbf{u}^{(0)} - \mathbf{u}^* \|_p \tag{3.8}$$

$$q \equiv \max (| 1 - \beta m |, \ | 1 - \beta M |) < 1$$

For the proof we examine the identity

$$\int_V p_{ij}(\mathbf{u})\, \varepsilon_{ij}(\mathbf{v})\, dV = \int_V p_{ij}(\mathbf{u})\, \varepsilon_{ij}(\mathbf{v})\, dV - \tag{3.9}$$

$$\beta \left[\int_V \sigma_{ij}(\mathbf{u})\, \varepsilon_{ij}(\mathbf{v})\, dV - A^e(\mathbf{v}) \right]$$

On the left in (3.9) is the scalar product $(\mathbf{u}, \mathbf{v})_p$. The right side is a linear functional of $\mathbf{v}$ according to (3.3). Using the imbedding theorem of Sobolev [5], it can be established that for this it is necessary that condition (3.5) be satisfied. Then, by the Riesz theorem, this functional can be represented as the scalar product $(\mathbf{u}', \mathbf{v})$, where $\mathbf{u}' \in \Pi$. Therefore, a certain operator Q sets each function $\mathbf{u} \in \Pi$ in correspondence to the function $\mathbf{u}' \in \Pi$. Hence, finding the generalized solution of the problem A converges to solving the operator equation

$$\mathbf{u} = Q\mathbf{u}$$

From (3.9), by using the equality (2.11) and condition (3.3) we have for two vector fields $\mathbf{u}_1$ and $\mathbf{u}_2$ and their difference $\mathbf{w} = \mathbf{u}_2 - \mathbf{u}_1$

$$| (Q\mathbf{u}_2 - Q\mathbf{u}_1, \mathbf{w})_p | = | (\mathbf{w}, \mathbf{w})_p - \tag{3.10}$$

$$\beta \int_V [\sigma_{ij}(\mathbf{u}_2) - \sigma_{ij}(\mathbf{u}_1)]\, w_{ij}\, dV \,| \leqslant q \| \mathbf{w} \|_p^2$$

$$w_{ij} \equiv \varepsilon_{ij}(\mathbf{u}_2) - \varepsilon_{ij}(\mathbf{u}_1)$$

where q is determined from the second relationship in (3.8). Hence

$$| 1 - \beta m | \geqslant | 1 - \beta M |, \quad 0 < \beta \leqslant 2/(m + M)$$

$$| 1 - \beta M | \geqslant | 1 - \beta m |, \quad 2/(m + M) \leqslant \beta \leqslant 2/M$$

Consequently, the condition $q < 1$ is satisfied for $0 < \beta \leqslant 2/M$ and the inequality (3.15) is satisfied if

$$\| Q\mathbf{u}_2 - Q\mathbf{u}_1 \|_p \leqslant q \| \mathbf{u}_2 - \mathbf{u}_1 \|_p \tag{3.11}$$

Let us note that the least value $q = (M - m)/(M + m)$ of the quantity q is reached for $\beta = 2/(m + M)$. Let us also note that the value of β can change in each iteration step so that $\beta^{(n)} \in (0, 2/M]$.

It follows from the inequality (3.11) that the operator Q performs a compression mapping in Π [2]. Furthermore, we have

$$(Q\mathbf{u} - Q\mathbf{u}^{(0)}, \mathbf{v})_p = (Q\mathbf{u} - Q\mathbf{u}^{(0)}, \mathbf{v})_p + (Q\mathbf{u}^{(0)} - \mathbf{u}^{(0)}, \mathbf{v})_p \tag{3.12}$$

But there follows from the identity (3.9)

$$(Q\mathbf{u}^{(0)} - \mathbf{u}^{(0)}, \mathbf{v})_p = \beta \int_V [\sigma_{ij}(\mathbf{u}^{(0)}) - p_{ij}(\mathbf{u}^{(0)})]\, \varepsilon_{ij}(\mathbf{v})\, dV \tag{3.13}$$

Applying condition (3.5) to (3.13) and setting $\mathbf{v} = \mathbf{u} - \mathbf{u}^{(0)}$ in (3.12), we obtain

$$\| Q\mathbf{u} - \mathbf{u}^{(0)} \|_p \leqslant (q + \beta m)\, r \leqslant r$$

i.e., the operator Q which performs the compression mapping, does not extract any point from the neighborhood (3.7). Hence, according to the principle of compression mappings [2], there exists a generalized solution of problem A. Its uniqueness follows from the appropriate application of Theorem 2.3.

It follows from (3.11) that the successive approximations converge as a geometric progression with the denominator q. The corollary of (3.11)

$$\| \mathbf{u}^{(n)} - \mathbf{u}^* \|_p \leqslant \frac{q^n}{1-q} \| \mathbf{u}^{(1)} - \mathbf{u}^{(0)} \|_p$$

has a more practical value. The theorem is proved.

The convergence of $L(\mathbf{u}^{(n)})$ to $L(\mathbf{u}^*)$ follows from the convergence of $\mathbf{u}^{(n)}$ to $\mathbf{u}^*$ [6]. In order to obtain a more rapid convergence of the iteration process than a geometric progression, a constraint should be applied on the second functional derivatives of the governing relations (1.1). For an arbitrary symmetric tensor $\mathbf{h}$ let the following inequality be valid

$$\left| \left[\frac{\partial^2 \sigma_{ij}}{\partial \varepsilon_{kl}\, \partial \varepsilon_{mn}} h_{kl} h_{mn} \right] h_{ij} \right| \leqslant \Lambda\, (h_{ij} h_{ij})^{3/2}, \quad \Lambda > 0 \tag{3.14}$$

Moreover, let us assume that the space Π_1 with the scalar product introduced

$$(\mathbf{u}, \mathbf{v})_1 \equiv \int_V \left[\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \varepsilon_{kl}(\mathbf{u}) \right] \varepsilon_{kl}(\mathbf{v})\, dV$$

is a Hilbert space for the functions $\mathbf{u} \in U_0$ defined in a finite domain V. Then the following theorem is valid

T h e o r e m 3 . 2. ("Rapidly converging" method). Let the operator P_{ij} of (3.1) have the form

$$p_{ij}(\mathbf{h}) \equiv \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} h_{kl}$$

and let a unique generalized solution of the appropriate problem $\mathbf{A}_p$ exist. Let the inequalities (3.14) and the inequality

$$m_1 h_{ij} h_{ij} \leqslant \left[\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} h_{kl} \right] h_{ij} \leqslant M_1 h_{ij} h_{ij}, \quad 0 < m_1 \leqslant M_1$$

be satisfied. Moreover, let a be a positive number such that

$$\int_V [\sigma_{ij}(\mathbf{u}^{(0)}) - p_{ij}(\mathbf{u}^{(0)})]\, \varepsilon_{ij}(\mathbf{u}^{(0)})\, dV \leqslant m_1 a \int_V \varepsilon_{ij}(\mathbf{u}^{(0)})\, \varepsilon_{ij}(\mathbf{u}^{(0)})\, dV$$

Then there is a number α

$$0 < \alpha \leqslant 1$$

such that the problem A has a uniquegeneralizedsolution $\mathbf{u}^*$ in the neighborhood

$$\| \mathbf{u}^{(0)} - \mathbf{u}^* \|_1 \leqslant r_0$$

if the inequality

$$q \leqslant a^{-\alpha} C; \quad q \equiv {}^3/_2 \frac{\Lambda}{m_1} V^{-\alpha/2}, \quad C \equiv \alpha (1 + \alpha)^{-(1+\alpha)/\alpha}$$

is satisfied, where r_0 is the least root of the equation

$$qr^{1+\alpha} - r + a = 0$$

Starting with $\mathbf{u}^{(0)}$ a successive approximations process converges to this solution for $\beta = 1$, where

$$\| \mathbf{u}^{(n)} - \mathbf{u}^* \|_1 \leqslant C_1 \delta^{(1+\alpha)^n}$$

$$\delta \equiv C^{1/\alpha}, \quad C_1 \equiv \frac{a}{\delta (1 - \delta)}$$

The proof of this theorem is presented in [6]. Certain examples of applying Theorems 3.1. and 3.2 to specific viscoelastic and elastic-plastic media are constructed there.

4. Now, let us assume that the operator relations (1.1) are uniquely solvable for the strain

$$\varepsilon_{ij} = G_{ij}(\sigma) \tag{4.1}$$

As is known, the Saint-Venant compatibility equations which make the symmetric incompatibility tensor η vanish

$$\eta_{ij} \equiv \epsilon_{ikl} \epsilon_{jmn} \varepsilon_{kn,\, lm} = 0 \quad (\eta \equiv \operatorname{Ink} \varepsilon = 0) \tag{4.2}$$

are the integrability conditions for the system of differential equations (1.2) in the displacements.

For a simply connected domain V the conditions (4.2) are necessary and sufficient for unique solvability (1.2) in the displacements, for instance, in the form proposed by Cesaro [7]

$$u_i(\mathbf{x}) = u_i' - (x_j - x_j')\, \omega_{ij}' + \int_{M'(\mathbf{x}')}^{M(\mathbf{x})} [\varepsilon_{im} + (x_j - \xi_j)(\varepsilon_{mi,\, j} - \varepsilon_{mj,\, i})] d\xi_m \tag{4.3}$$

where u_i' and ω_{ij}' are known values of the displacements and rotations at some point M' $(\mathbf{x}')$ of the domain V. Therefore, if conditions (4.2) are satisfied then there exists a vector $\mathbf{u}$ for which the Cauchy relationships are valid. According to (4.2), the deviator and the spherical part of the tensor η evidently vanish. Hence, their comination also vanishes:

$$\Delta \varepsilon_{ij} + \theta_{,\, ij} - \varepsilon_{ik,\, kj} - \varepsilon_{jk,\, ki} + \xi_{ij} (\varepsilon_{kl,\, kl} - \Delta \theta) = 0 \tag{4.4}$$

where ξ is an arbitrary symmetric constant tensor.

Substituting (4. 1) into (4. 2) and (4. 3), we obtain a system of six equations in the stress tensor components and boundary conditions

$$\eta_{ij}(\sigma) = 0 \tag{4.5}$$
$$u_i(\sigma)|_{\Sigma_1} = u_i^\circ, \quad \sigma_{ij}n_j|_{\Sigma_2} = S_i^\circ$$

Therefore, a formulation of the quasistatic (static) problem of the mechanics of a deformable solid is given in stresses (problem B) by the relations (1. 3), (4. 2), (4. 1) or (1. 3) and (4. 5). The formulations of problems A and B are evidently mutually equivalent.

Let us multiply (1. 2) scalarly by the tensor $\tau \in T_0$ and let us integrate over the volume V. Then by using the Ostrogradskii—Gauss theorem and condition (1. 1) we obtain

$$\int_V \varepsilon_{ij}\tau_{ij}\,dV = A_{\Sigma_1}(u^\circ) \tag{4.6}$$

Let us call the tensor $\sigma \in T$ for which the governing relations (4. 1) are valid and which satisfies the identities (4. 6) for every sufficiently smooth tensor function $\tau \in T_0$ the generalized solution of the problem B. In other words, the tensor-function $\sigma \in T$ satisfying the integral identity

$$\int_V \varepsilon_{ij}(\sigma)\,\tau_{ij}\,dV = A_{\Sigma_1}(\mathbf{u}^\circ) \tag{4.7}$$

for every smooth tensor function $\tau \in T_0$ is called a generalized solution of the problem B. It has been shown above that the solution of problem B is also its generalized solution.

T h e o r e m 4. 1. If the generalized solution is sufficiently smooth, then it is a solution of problem B.

Indeed, the solution of problem B in a simply-connected domain should satisfy conditions (1. 3), (4. 2), (4. 1). By the definition of the generalized solution, (1. 3), the relations (4. 1), and the second of the boundary conditions in (1. 3) are satisfied. Introducing a system of smooth functions $\varkappa_i(\mathbf{x}), \mathbf{x} \in V$ and $\varkappa_i'(\mathbf{y}), \mathbf{y} \in \Sigma_2$ (generalized Lagrange multipliers), we can write

$$\int_{\Sigma_1} \tau_{ij}n_ju_i^\circ\,d\Sigma - \int_V \varepsilon_{ij}\tau_{ij}\,dV - \int_V \varkappa_i(\sigma_{ij,j} + X_i)\,dV - \int_{\Sigma_2} \varkappa_i'(\sigma_{ij}n_j - S_i^\circ)\,d\Sigma = 0 \tag{4.8}$$

Applying the Ostrogradskii—Gauss theorem to (4. 8), we obtain because of the arbitrariness of the field $\tau \in T_0$

$$\varepsilon_{ij} = {}^1/_2\,(\varkappa_{i,j} + \varkappa_{j,i}), \quad \varkappa_i|_{\Sigma_1} = u_i^\circ \tag{4.9}$$

For a continuous field $\varkappa$ to exist, it is necessary and sufficient to satisfy conditions (4. 2), where compliance with the first of the boundary conditions (1. 3) follows from (4. 9).

Now, let us assume that the strain tensor is potential, i. e., a scalar operator of the stresses $w(\sigma)$ exists such that

$$\varepsilon_{ij} = G_{ij}(\sigma) = \partial w(\sigma)/\partial\sigma_{ij} \tag{4.10}$$

In this case, the so-called "Castiglianian" K can be introduced by means of the formula

$$K(\sigma) \equiv -\varphi(\sigma) + A_{\Sigma_1}(\mathbf{u}^\circ), \quad \varphi(\sigma) \equiv \int_V w\,dV$$

Then, the identity (4.7) can evidently be written in the form

$$DK\{\sigma, \tau\} = 0$$

Therefore, the problem of seeking the generalized solution of the problem B is equivalent to seeking the "stationary point" of the Castiglianian $K(\sigma)$.

Without examining the conditions for the existence of a maximum of the Castiglianian, let us note the following. According to the generalized Legendre transformation, we set the operator $w(\sigma)$ for which the relations (4.10) are satisfied in such a manner that the identity

$$W + w - \sigma_{ij}\varepsilon_{ij} = \text{const} \tag{4.11}$$

is satisfied, in correspondence to the operator $W(\varepsilon)$ for which the relations (2.3) are valid. Hence, if $W(0) = 0$ and $w(0) = 0$, then the constant in the right side of (4.11) equals zero.

Theorem 4.2. The Lagrangian agrees with the Castiglianian at equilibrium. Indeed, let us examine the identity (1.5). Using the relationship (4.11) we obtain

$$L(\mathbf{u}^*) = K(\sigma^*)$$

where $\mathbf{u}^*$, σ^* are solutions of problems A and B, respectively.

5. Now, let us give a new formulation of the problem of the mechanics of a deformable solid in terms of stresses. To do this, we apply the operation Def to the equilibrium equations (1.3)

$$S_{ij} \equiv {}^1/_2 (\sigma_{ik,kj} + \sigma_{jk,ki} + X_{i,j} + X_{j,i}) \tag{5.1}$$

By using the compatibility equations (4.4), written in stresses and the relationships (5.1), we form the equation

$$\Delta\varepsilon_{ij}(\sigma) + \theta_{,ij}(\sigma) - \varepsilon_{ik,kj}(\sigma) - \varepsilon_{jk,ki}(\sigma) + \xi_{ij}(\varepsilon_{kl,kl}(\sigma) - \Delta\theta(\sigma)) + Q_{ij}(S) + (\xi_{ij} - \delta_{ij})Q_{mm}(S) = 0 \tag{5.2}$$

where Q_{ij} are components of the symmetric tensor operator of the tensor (5.1). We write the equilibrium equations for points on the surface of the body Σ

$$(\sigma_{ij,j} + X_i)|_\Sigma = 0 \tag{5.3}$$

Therefore, we have six equations (5.2) in six independent components of the stress tensor σ and six boundary conditions (4.5) and (5.3). This is indeed a new formulation of the problem of mechanics of a deformable solid in terms of stresses (the problem B) [8].

Theorem 5.1. If the operators of the governing relations (1.1) and (4.1) are mutually inverse, the problems A and B are mutually equivalent.

It has been shown above that the formulation of problem B follows from the formulation of problem A. Now, let a formulation of problem B be given. Let (5.2) be convoluted with the unit tensor δ_{ij}

$$(2 - \xi_{mm})[\Delta\theta(\sigma) - \varepsilon_{kl,kl}(\sigma) - Q_{mm}(S)] \tag{5.4}$$

We now apply the operation Div to (5.2)

$$(\delta_{ij} - \xi_{ij})[\Delta\theta(\sigma) - \varepsilon_{kl,kl}(\sigma) - Q_{mm}(S)]_{,j} + Q_{ij,j}(S) = 0 \tag{5.5}$$

It follows from (5.4) and (5.5) that for $\xi_{mm} \neq 2$

$$Q_{ij,j}(S) = 0 \tag{5.6}$$

If the operator Q is such that its functional derivatives with respect to the tensor (5.1) satisfies conditions (2.9), then (1.3) follows from (5.6) and (5.3). The validity of the compatibility conditions (4.4) hence follows, meaning (4.2) is also valid. Hence, there exists a vector **u** for which the relations (1.2) are valid. The theorem is proved.

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